HW IV

Let $x_n \in \mathbb{R}\ \forall n$ and $S_n := x_1 + \cdots + x_n$; let $r \in (0,1)$.

1. Show $\lim_N \sum_{n=1}^{N} r^n$ exists (in $\mathbb{R}$) for each of the following methods:
   (i) use the Bounded Monotone Conv. Th.
   (ii) $\left(\sum_{n=1}^{N} r^n : N \in \mathbb{N}\right)$ is a Cauchy seq.

2. Let (x_n) be a contractive seq with rate r :
$$|x_{n+1} - x_n| \leq r\,|x_n - x_{n-1}| \quad \forall n \in \mathbb{N}, n>1. \quad (1)$$
Show that
$$|x_{n+1} - x_n| \leq r^{n-1}\,|x_2 - x_1| \quad \forall n \in \mathbb{N} \quad (2)$$
and that
$$|x_{n+k} - x_n| \leq \frac{r^{n-1}}{1-r}\,|x_2 - x_1| \quad \forall n, k \in \mathbb{N}. \quad (3)$$
Using each of the following suggestions, show that (x_n) converges.
   (i) $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ is absolutely convergent and so convergent (hence $\lim_n (x_{n+1} - x_1)$ exists in $\mathbb{R}$); $\therefore \lim_n x_n$ exists
   (ii) By (3), (x_n) is Cauchy.

3. Let $x_1 := 1$, $x_2 := 2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})\ \forall n > 2$. Show by Q2 that (x_n) converges. But it is quite difficult to find its limit: show by MI that
$$x_{2n-1} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^3 + \cdots + \left(\tfrac{1}{2}\right)^{2n-3} \longrightarrow 1 + \tfrac{1}{2}\left(\frac{1}{1-1/4}\right) = 5/3$$
$$x_{2n} = 2 - \frac{1}{2^2} - \frac{1}{2^4} - \cdots - \left(\tfrac{1}{2}\right)^{2n-2} \longrightarrow 2 - \tfrac{1}{4}\left(\frac{1}{1-1/4}\right) = 5/3$$
Hence $\lim_n x_n = 5/3$ (Why?)

4. Let $0 \leq x_n \downarrow_n$ (i.e. $x_n \geq x_{n+1}\ \forall n$) and let $t_n := \sum_{K=1}^{n} 2^K x_{2^K}$. Then, in "grouping",
$$x_1 + (x_2 + x_3) + (x_4 + \cdots + x_{2^3-1}) + (x_{2^3} + \cdots + x_{2^4-1}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1})$$
$$\leq x_1 + (x_2 \times 2) + (x_{2^2} \times 2^2) + (x_{2^3} \times 2^3) + \cdots (x_{2^{n-1}} \times 2^{n-1}) = \sum_{K=1}^{n} x_{2^{K-1}} \cdot 2^{K-1}. \quad (1)$$
Also, in different grouping,
$$x_1 + x_2 + (x_3 + x_4) + (x_5 + \cdots + x_8) + \cdots + (x_{2^{n-1}+1} + \cdots x_{2^n})$$
$$\geq x_1 + x_2 + 2 \cdot x_{2^2} + 2^2 x_{2^3} + \cdots + 2^{n-1} x_{2^n} \geq x_1 + 2^{-1}\left(\sum_{K=1}^{n} x_{2^K} \cdot 2^K\right) \quad (2)$$

4 (continue). Thus $s_{2^n-1} \leq t_n$ and $s_{2^n} \geq x_1 + \frac{1}{2} t_n \ \forall n \geq 2$

and so $\sum_{n=1}^{\infty} x_n$ conv iff its "condensation" series

$\sum_{n=1}^{\infty} 2^n x_{2^n}$ conv. (Cauchy condensation theorem).

Remark. The method of different ways of grouping is also used to deal with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ (div), and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (conv) for $p > 1$.

(Can you do them?)

5. Let $a > 0$ and $z_1 > 0$. Let ξ, M be defined by

$$\xi := \max\{1, z_1\} \quad \text{and} \quad M = \sqrt{a} + \xi$$

(so $\sqrt{a + \cdot} \leq \sqrt{a} + \cdot$). Let (z_n) be defined by

$$z_n = \sqrt{a + z_{n-1}} \quad \forall n > 1.$$

Show, by MI, that each $z_n \leq M$ and $(z_n) \uparrow$ or $\downarrow$ (depending on $z_1 \leq z_2$ or $z_1 \geq z_2$). Why the seq converges and to what (which root of an equation)?

6*. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2}(x_n + 3) \ \forall n$. Using each of the following suggestions to show the convergence of (x_n) [and find the limit]:

(i) (x_n) is $\uparrow$ and bounded by 3

(ii) (x_n) is contractive: $|x_{n+2} - x_{n+1}| \leq r |x_{n+1} - x_n|$ with appropriate $r \in (0,1)$.

7*. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above with $s := \sup A$. Suppose $s \notin A$. Show that $\exists\, (a_n)$ strictly increasing such that $\lim a_n = s$.

8*. Q10 of p74. (regarding limit superior/inferior).